WEAK SPECIFICATION PROPERTIES AND LARGE DEVIATIONS FOR NON-ADDITIVE POTENTIALS

PAULO VARANDAS AND YUN ZHAO

Abstract. We obtain large deviation bounds for the measure of deviation sets associated to asymptotically additive and sub-additive potentials under some weak specification properties. In particular a large deviation principle is obtained in the case of uniformly hyperbolic dynamical systems. Some examples in connection with the convergence of Lyapunov exponents are given.

1. Introduction

The purpose of the theory of large deviations is to study the rates of convergence of sequences of random variables to some limit distribution. Some applications of these ideas into the realm of Dynamical Systems have been particularly useful to estimate the velocity at which time averages of typical points of ergodic invariant measures converge to the space average as guaranteed by Birkhoff's ergodic theorem. More precisely, given a continuous transformation f on a compact metric space M and a reference measure ν, one interesting question is to obtain sharp estimates for the ν-measure of the deviation sets $\{x \in M : \frac{1}{n}\sum_{j=0}^{n-1} g(f^j(x)) > c\}$ for all continuous functions $g : M \to \mathbb{R}$ and real numbers c. We refer the reader to [You90, Kif90, KN91, EKW94, PS05, AP06, MN08, RBY08, CRL98, Yu07, Mel09, PS09, Co09, Chu11, Va12] and the references therein for an account on recent large deviations results.

However, since many relevant quantities in dynamical systems arise from non-additive sequences (e.g. larger Lyapunov exponent for dynamics in dimension larger than one) through Kingman's ergodic theorem it is also a fundamental to study the deviation sets $\{x \in M : \varphi_n(x) > cn\}$ with respect to some not necessarily additive sequences $\Phi = \{\varphi_n\}_n$ of continuous functions. Problems involving scaled limits of such a sequence and occurring in dimension theory or multifractal analysis play important roles in the theory dynamical systems. Inspired by the pioneering work of Young [You90] our purpose in this direction is to provide sharp large deviations estimates for a wide class of non-additive sequences of continuous potentials. Our approach uses ideas from the non-additive thermodynamical formalism and the reference measures we are mostly interested are Gibbs measures for some regular family of potentials. For that reason we will often require more regularity in the family of potentials that lead to the equilibrium measures rather than on the family of observables with respect to which we measure deviations. Some recent results

[0]2000 *Mathematics Subject classification*: 37A30, 37D35, 37H15, 60F10
Date: February 27, 2013.
Key words and phrases. Large deviations, asymptotically additive and sub-additive potentials, non-additive thermodynamical formalism, weak Gibbs measure.

considering the thermodynamical formalism of almost additive or sub-additive sequences of potentials include [FL02, Ba06, Mu06, IY11, FK12], whereas in all cases the authors proved that there exists a unique equilibrium state μ_Φ and it is absolutely continuous with respect to a Gibbs measure ν_Φ with density bounded away from zero and infinity. Building over [Ba96], Méson and Vericat [MV09] obtained bounds for large deviations processes for a family of sub-additive potentials $\Phi = \{\varphi_n\}$, namely those such that $\varphi_n - \varphi_{n-1} \circ f$ converge uniformly. Our purpose here it to extend the theory beyond the hyperbolic context and to consider a broad class of sub-additive and asymptotically additive sequences of potentials.

Furthermore, to be able to deal with some non-uniformly hyperbolic dynamical systems or dynamical systems that admit mistakes a key point is to obtain some weak specification property. In fact, as a physical process evolves it is natural for the evolving process to change or produce some errors in the evaluation of orbits. However, a self-adaptable system should decrease errors over time. This is a motivation for this study the large deviations for non-additive potentials when the systems admits mistakes. In fact, if on the one hand it is known that stable specification property for diffeomorphisms coincides with uniformly hyperbolic dynamical systems (see [SSY09]) on the other hand not only the notions of specification and topologically mixing coincide for every one-dimensional *continuous* mapping (see [Blo83]) as weaker specification properties hold in the presence of nonuniform hyperbolicity. Roughly, one proves that the set of points whose sequences values remain far from the space average with respect to the equilibrium measure decrease exponentially fast. In particular we obtain a large deviations principle for non-additive sequences in the uniformly hyperbolic setting. As important applications we estimate the rate of convergence of the maximal Lyapunov exponent for some families of linear cocycles and and local diffeomorphisms that satisfy some cone condition. We refer the reader to Section 4 for precise statements and details.

The remainder of this paper is organized as follows. In section 2, we present some definitions and fundamental notions necessary to state our results. In Section 3 we present statements of the main results in this paper. Some examples are given in Section 4 while the proofs of the main results are given in section 5. Finally in the Appendix A we estimate the measure of mistake dynamical balls in the uniformly expanding setting.

2. Preliminaries

Throughout this paper, (M, f) denotes a continuous dynamical systems in the sense that $f : M \to M$ is a (piecewise) continuous transformation on the compact metric space M with a metric d. Invariant Borel probability measures are associated with (M, f). Let $\mathcal{M}_f$ and $\mathcal{E}_f$ denote the space of f-invariant Borel probability measures and the set of f-invariant ergodic Borel probability measures, respectively.

2.1. Specification properties.

Specification properties are very useful to obtain existence of equilibrium states as well as large deviation principles. Here we introduce and discuss some different notions.

Definition 2.1. *We say that a map f satisfies the* specification property *if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \dots, x_k$, and any sequence of positive integers $n_1, \dots, n_k$ and*

$p_1, \dots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point x in M such that

$$d\Big(f^j(x), f^j(x_1)\Big) \leq \varepsilon, \quad \forall\, 0 \leq j \leq n_1$$

and

$$d\Big(f^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\Big) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

The previous notion is slightly weaker than the one introduced by Bowen [Bow71], that requires that any finite sequence of pieces of orbit is well approximated by periodic orbits. Despite robust specification property for diffeomorphisms is satisfied only by uniformly hyperbolic dynamical systems (see Sakai, Sumi and Yamamoto [SSY09]) we know by Blokh [Blo83] that the notions of specification and topologically mixing coincide for every one-dimensional *continuous* mapping. This is no longer true if the one-dimensional map fails to be continuous (see e.g. [Buz97]).

To define other weak form of specification we first recall the definitions of mistake function and mistake dynamical balls which are due to Thompson [Th10], Pfister and Sullivan [PS07]. Given $\varepsilon_0 > 0$ the function $g : \mathbb{N} \times (0, \varepsilon_0] \to \mathbb{N}$ is called a *mistake function* if for all $\varepsilon \in (0, \varepsilon_0]$ and all $n \in \mathbb{N}$, $g(n, \varepsilon) \leq g(n+1, \varepsilon)$ and $\lim_n g(n, \varepsilon)/n = 0$. By a slight abuse of notation we set $g(n, \varepsilon) = g(n, \varepsilon_0)$ for every $\varepsilon > \varepsilon_0$. Moreover, for any subset of integers $\Lambda \subset [0, N]$, we will use the family of distances in the metric space M given by $d_\Lambda(x, y) = \max\{d(f^i x, f^i y) : i \in \Lambda\}$ and consider the balls $B_\Lambda(x, \varepsilon) = \{y \in M : d_\Lambda(x, y) < \varepsilon\}$. Hence we can now consider mistake dynamical balls. Given a mistake function g, $\varepsilon > 0$ and $n \geq 1$, the $(n, \varepsilon)-$*mistake dynamical ball* $B_n(g; x, \varepsilon)$ of radius ε and length n associated to g is defined by

$$\begin{aligned} B_n(g; x, \varepsilon) &= \{y \in M \mid y \in B_\Lambda(x, \varepsilon) \text{ for some } \Lambda \in I(g; n, \varepsilon)\} \\ &= \bigcup_{\Lambda \in I(g;n,\varepsilon)} B_\Lambda(x, \varepsilon) \end{aligned}$$

where $I(g; n, \varepsilon) = \{\Lambda \subset [0, n-1] \cap \mathbb{N} \mid \#\Lambda \geq n - g(n, \varepsilon)\}$. A set $F \subset Z$ is $(g; n, \varepsilon)-$separated for Z if for every $x, y \in F$ with $x \neq y$ implies $d_\Lambda(x, y) > \varepsilon, \forall \Lambda \in I(g; n, \varepsilon)$. The dual definition is as follows. A set $E \subset Z$ is $(g; n, \varepsilon)-$spanning for Z if for all $z \in Z$, there exists $x \in E$ and $\Lambda \in I(g; n, \varepsilon)$ such that $d_\Lambda(x, z) \leq \varepsilon$.

Definition 2.2. *Let g be a mistake function. We say that f satisfies the g-*almost specification property *if there exists $\varepsilon > 0$ and a positive integer $N(g, \varepsilon)$ such that the following property holds: for every $k \geq 1$, any points $x_1, \dots, x_k$, and any positive integers $n_1, \dots, n_k$ with $n_i \geq N(g, \varepsilon)$ it follows that*

$$\bigcap_{i=1}^{k} f^{-\sum_{j=0}^{i-1} n_j}(B_{n_i}(g; x_i, \varepsilon)) \neq \emptyset$$

where $n_0 = 0$.

The later property holds for all β-transformations (see [PS07, Th10]). In fact Thompson [Th10] introduced a more general property where the value ε is replaced by several values $\varepsilon_1, \dots, \varepsilon_k$. However this weaker notion is suitable for our purposes.

2.2. **Non-additive potentials.** Now we turn our attention for different notions of not necessary additive potentials. Let $C(M)$ denote the space of continuous functions from M to $\mathbb{R}$. A sequence $\Phi = \{\varphi_n\} \subset C(M)$ is a *sub-additive* (respectively sup-additive) sequence of potentials if $\varphi_{m+n} \le \varphi_m + \varphi_n \circ f^m$ (respectively $\varphi_{m+n} \ge \varphi_m + \varphi_n \circ f^m$) for every $m, n \ge 1$.

We say that the sequence $\Phi = \{\varphi_n\} \subset C(M)$ is an *almost additive* sequence of potentials if there exists a uniform constant $C > 0$ such that $\varphi_m + \varphi_n \circ f^m - C \le \varphi_{m+n} \le \varphi_m + \varphi_n \circ f^m + C$ for every $m, n \ge 1$. Finally, we say that $\Phi = \{\varphi_n\} \subset C(M)$ is an *asymptotically additive* potential on M if for any $\xi > 0$ there exists a continuous function φ_ξ such that

$$\limsup_{n\to\infty} \frac{1}{n} \|\varphi_n - S_n\varphi_\xi\| < \xi \tag{2.1}$$

where $S_n\varphi_\xi = \sum_{j=0}^{n-1} \varphi_\xi \circ f^j$ denotes the usual Birkhoff sum, and $||\cdot||$ is the sup norm on the Banach space $C(M)$. Let $\mathcal{A}$ denote the set of asymptotically additive potentials. The following result establishes the relation between these notions.

Proposition 2.1. *The following properties hold:*

1. *If $\Phi = \{\varphi_n\}$ is almost additive there exists $C > 0$ such that the sequence $\Phi_C = \{\varphi_n + C\}$ is sub-additive and $\Phi_{-C} = \{\varphi_n - C\}$ is sup-additive;*
2. *If $\Phi = \{\varphi_n\}$ is almost additive then it is asymptotically additive, and for any $\xi > 0$ there exists $k = k(\xi) \ge 1$ so that $\limsup_{n\to\infty} \frac{1}{n}\big\|\varphi_n - S_n\big(\frac{1}{k}\varphi_k\big)\big\| < \xi$.*

Proof. Part (1) is obvious from the definitions. Part (2) is contained in Proposition A.5 of [FH10] or Proposition 2.1 of [ZZC11]. □

In Example 4.6 one provides examples of sub-additive and sup-additive potentials that are almost additive, while Example 4.7 gives an example of sub-additive and sup-additive potentials that are asymptotically additive. It is not known if there are asymptotically additive potentials not sub-additive or almost additive.

By Kingman's subadditive ergodic theorem it follows that for every sub-additive potential $\Phi = \{\varphi_n\}$ and every f-invariant ergodic probability measure μ it holds

$$\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \inf_{n\ge1} \frac{1}{n} \int \varphi_n \, d\mu =: \mathcal{F}_*(\mu, \Phi), \qquad \text{for } \mu\text{-a.e. } x. \tag{2.2}$$

In fact, Feng and Huang [FH10] proved that the same property holds for asymptotically additive potentials and, consequently, for all almost additive ones. In that paper, Feng and Huang also proved that the map $\mu \mapsto \mathcal{F}_*(\mu, \Phi)$ is continuous (respectively upper semi-continuous) if Φ is asymptotically additive (respectively sub-additive).

2.3. **Non-additive topological pressure and equilibrium states.** Finally, we recall the notions of non-additive topological pressure. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ we consider the $(n, \varepsilon)-$dynamical balls $B_n(x, \varepsilon) := \{y \in M : d_n(x, y) < \varepsilon\}$, where $d_n(x, y) = \max_{0\le i<n} d(f^i x, f^i y)$. We say that a set $E \subset M$ is (n, ε)-*separated* if all distinct $x, y \in E$ satisfy $y \notin B_n(x, \varepsilon)$. If $\Phi = \{\varphi_n\}$ is a non-additive (respectively sub-additive, almost additive or asymptotically additive) family of potentials, the *topological pressure of f with respect to* Φ is defined by

$$P(f, \Phi) = \lim_{\varepsilon\to0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{E_n} \{Z_n(\Phi, E_n, \varepsilon)\}$$

where $Z_n(\Phi, E_n, \varepsilon) = \sum_{y \in E_n} \exp(\varphi_n(y))$ and the supremum is taken over all (n, ε)-separated sets. The following variational principle relates the non-additive pressure with the natural modifications of the measure-theoretic free energy. Just recall first a very useful formula to compute the metric entropy was given by Katok [Kat80, Theorem I.I] showing that if η is an f-invariant ergodic probability measure then

$$h_\eta(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta), \tag{2.3}$$

where $N(n, \varepsilon, \delta)$ is the minimum number of (n, ε)-dynamical balls necessary to cover a set of $\eta-$measure larger than δ. More generally, it was proven in [Th10] that for any mistake function g under the previous assumptions it also holds that

$$h_\eta(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(g; n, \varepsilon, \delta) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(g; n, \varepsilon, \delta), \tag{2.4}$$

where the term $N(g; n, \varepsilon, \delta)$ stands for the minimum number of (n, ε)-mistake dynamical balls necessary to cover a set of $\eta-$measure larger than δ. A generalization of formulas (2.3) and (2.4) to measure theoretic pressure was given in [CZC12] and [CHZ].

Theorem 2.1. *Let $f : M \to M$ be a continuous map on the compact metric space M, and $\Phi = \{\varphi_n\}$ a non-additive potential (respectively sub-additive, almost additive or asymptotically additive). Then*

$$P(f, \Phi) = \sup\{h_\mu(f) + \mathcal{F}_*(\mu, \Phi) : \mu \in \mathcal{M}_f,\ \mathcal{F}_*(\mu, \Phi) \neq -\infty\}.$$

We refer the reader to [CFH08, Ba06, Mu06, FH10] for the proof of this variational principle and details on topological pressure of non-additive potentials. An f-invariant probability measure μ that attains the supremum is called *equilibrium state* for f with respect to Φ. In many situations these arise as invariant measures absolutely continuous with respect to (weak) Gibbs measures, that we recall.

Definition 2.3. *Given a sequence of functions $\Phi = \{\varphi_n\}$ we say that a probability measure ν is a* weak Gibbs measure with respect to Φ on $\Lambda \subset M$ *if the set Λ has full ν-measure and there exists $\varepsilon_0 > 0$ such that for every $x \in \Lambda$ and $0 < \varepsilon < \varepsilon_0$ there exists a sequence of positive constants $(K_n)_{n \geq 1}$ (depending only on ε) satisfying $\lim_{n \to \infty} \frac{1}{n} \log K_n = 0$ and for every $n \geq 1$*

$$K_n^{-1} \leq \frac{\nu(B_n(x, \varepsilon))}{e^{-nP(f,\Phi) + \varphi_n(x)}} \leq K_n.$$

We say that ν is a Gibbs measure with respect to Φ *if there exists $K > 0$ such that the same property holds with $K_n = K$ independent of n.*

The previous notion of Gibbs measure is a generalization of the usual notion obtained in [Ba06, Mu06] in the uniformly hyperbolic setting. In the case of additive potentials, these weak Gibbs measures appear in dynamics with some non-uniform hyperbolicity as e.g. [VV10, Yu00]. Let us focus on some results concerning the existence of equilibrium states in the uniformly hyperbolic setting. Given a basic set Ω for an Axiom A diffeomorphism f it is known that every almost additive potential $\Phi = \{\varphi_n\}$ satisfying

(1) (bounded variation) $\exists A, \delta > 0 :\ \sup_{n \in \mathbb{N}} \gamma_n(\Phi, \delta) \leq A$,

where $\gamma_n(\Phi,\delta)=\sup\{|\varphi_n(y)-\varphi_n(z)| : \, y,z\in B_n(x,\delta)\}$, admit a unique equilibrium state μ_Φ which coincides with the Gibbs measure w.r.t. Φ (see [Ba06, Mu06] for the proof). This condition above was introduced by Bowen [Bow74] to obtain uniqueness of equilibrium states for expansive maps with the specification property. We say that a sequence of continuous functions $\Psi=\{\psi_n\}$ satisfy *weak Bowen condition* if there exist $\delta>0$ and a sequence of positive real numbers $\{a_n\}_n$ such that $\limsup\limits_{n\to\infty}\frac{a_n}{n}=0$ and

$$\gamma_n(\Phi,\delta)\le a_n, \qquad \text{for all } n\ge 1. \tag{2.5}$$

3. Statement of the results

Here we state our main results of this paper. The first one is a modified Brin-Katok local entropy formula for dynamical systems when some errors are admissible. We prove that the exponential decreasing rate of the measure of the mistake dynamical ball is equal to the measure-theoretic entropy.

Proposition A. *Given an f-invariant ergodic measure μ and a mistake function g, the following limits*

$$\underline{h}_\mu(g;f,x)=\lim_{\varepsilon\to 0}\liminf_{n\to\infty}-\frac{1}{n}\log\mu(B_n(g;x,\varepsilon))$$

and

$$\overline{h}_\mu(g;f,x)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}-\frac{1}{n}\log\mu(B_n(g;x,\varepsilon))$$

exist for μ-almost every x and coincide with the measure theoretic entropy $h_\mu(f)$.

Let us mention that, although expected, the proof of the later formulas cannot follow the original strategy of Brin and Katok. Notice that the mistake dynamical balls $B_n(g;x,\varepsilon)$ take into account not only the size n as the amount of allowed mistakes $g(n,\varepsilon)$. In particular, the mistake dynamical balls $B_n(g;x,\varepsilon)$ may not even satisfy the inclusion $B_{n+1}(g;x,\varepsilon)\subset B_n(g;x,\varepsilon)$, e.g, if $g(n,\varepsilon)$ is much larger than $g(n-1,\varepsilon)$. Since this fact is not standard an estimation on the measure of mistake dynamical balls for uniformly expanding maps is given in the Appendix A. Furthermore, the ergodicity assumption in Proposition A is not crucial. Given $\mu\in\mathcal{M}_f$ by ergodic decomposition theorem we know that μ can be decomposed as a convex combination of ergodic measures, $\mu=\int\mu_x \mathrm{d}\mu(x)$. Applying Proposition A to each ergodic component μ_x and using $h_\mu(f)=\int h_{\mu_x}(f)\mathrm{d}\mu(x)$ we obtain:

Corollary 3.1. *Given any $\mu\in\mathcal{M}_f$, the limits $\underline{h}_\mu(g;f,x)$ and $\overline{h}_\mu(g;f,x)$ do exist for μ-almost every x and the measure theoretic entropy $h_\mu(f)$ satisfies*

$$h_\mu(f)=\int\underline{h}_\mu(g;f,x)\mathrm{d}\mu(x)=\int\overline{h}_\mu(g;f,x)\mathrm{d}\mu(x).$$

Large deviation bounds for asymptotically additive observables. We are also interested to study the rate of convergence at Kingman's sub-additive theorem. More precisely, given a Borel probability measure m on the space M, we study the rate at which the m-measure of the sets

$$B(n)=\Big\{x\in M\colon \Big|\frac{1}{n}\varphi_n(x)-\mathcal{F}_*(\mu,\Phi)\Big|>c\Big\}$$

goes to zero as n tends to infinite, with respect to our reference and not necessarily invariant probability measure m. Given a mistake function g and a Borel probability ν we define

$$h_m(g; f, x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(B_n(g; , x, \varepsilon))$$

and $h_m(g; f, \nu) = \nu - \text{ess sup}\, h_m(g; f, x)$. It follows from Proposition A that we have $h_\nu(g; f, x) = h_\nu(f)$ for $\nu-$a.e. x and every $\nu \in \mathcal{E}_f$. To provide more precise bounds for $h_m(g; f, x)$ we introduce two sets of functions as follows.

Given a constant K and a mistake function g, define $\mathcal{V}_K^+(g)$ as the set of sequences $\Phi \in \mathcal{A}$ for which there exists $\varepsilon_0 > 0$ and a set Υ of full $m-$measure such that the following property holds: for all $0 < \varepsilon < \varepsilon_0$ there are constants C_n (depending only on ε) so that $\lim_{n\to\infty} \frac{1}{n} \log C_n = 0$ and

$$m(B_n(g; x, \varepsilon)) \leq C_n \exp(-nK + \varphi_n(x)), \quad \forall x \in \Upsilon \text{ and } n \geq 1.$$

We also consider the set $\mathcal{V}_K^-(g)$ as the set of sequences $\Phi \in \mathcal{A}$ for which there exists $\varepsilon_0 > 0$ and a set Υ of full $m-$measure such that the following property holds: for all $0 < \varepsilon < \varepsilon_0$ there are constants C_n so that $\lim_{n\to\infty} \frac{1}{n} \log C_n = 0$ and

$$m(B_n(g; x, \varepsilon)) \geq C_n \exp(-nK + \varphi_n(x)), \quad \forall x \in \Upsilon \text{ and } n \geq 1.$$

Observe that both classes of asymptotically potentials $\mathcal{V}_K^+(g)$, $\mathcal{V}_K^-(g)$ depend on the mistake function g. For the mistake function $g \equiv 0$ we will simply write $\mathcal{V}_K^+$ and $\mathcal{V}_K^-$ respectively. We shall refer to the constants C_n above as tempered constants. Finally, for any not necessarily additive sequence of observables $\Phi = \{\varphi_n\}$ and $E \subset \mathbb{R}$ define

$$\overline{R}_m(\Phi, E) = \limsup_{n\to\infty} \frac{1}{n} \log m\left(\left\{x \in M : \frac{1}{n}\varphi_n(x) \in E\right\}\right)$$

and

$$\underline{R}_m(\Phi, E) = \liminf_{n\to\infty} \frac{1}{n} \log m\left(\left\{x \in M : \frac{1}{n}\varphi_n(x) \in E\right\}\right).$$

The following abstract results generalize [You90, Theorem A] to the case of asymptotically additive potentials under some mistake dynamical systems.

Theorem A. *Assume $h_{top}(f) < \infty$. Then for each $\Phi \in \mathcal{A}$, $c \in \mathbb{R}$ and mistake function g the following holds:*

1. $\underline{R}_m(\Phi, (c, \infty)) \geq \sup\{h_\nu(f) - h_m(g; f, \nu) : \nu \in \mathcal{E}_f,\ \mathcal{F}_*(\nu, \Phi) > c\}$*;*
2. *For each $\Psi \in \mathcal{V}_K^+(g)$, we have*
$$\overline{R}_m(\Phi, [c, \infty)) \leq \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\}$$
where the supremum is taken over all $\nu \in \mathcal{M}_f$ satisfying $\mathcal{F}_*(\nu, \Phi) \geq c$;
3. *For each $\Psi \in \mathcal{V}_K^-(g)$, we have*
$$\underline{R}_m(\Phi, (c, \infty)) \geq \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\}$$
where the supremum is taken over all $\nu \in \mathcal{E}_f$ satisfying $\mathcal{F}_*(\nu, \Phi) > c$ and $\nu(\Upsilon) = 1$;
4. *Assume f satisfies the g-almost specification property. Then, given $\Psi \in \mathcal{V}_K^-$*
$$\underline{R}_m(\Phi, (c, \infty)) \geq \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\}$$
where the supremum is taken over all $\nu \in \mathcal{M}_f$ satisfying $\mathcal{F}_*(\nu, \Phi) > c$ and $\nu(\Upsilon) = 1$.

The previous result has particularly interesting applications to weak Gibbs measures obtained in thermodynamical formalism as we now describe.

Theorem B. *Assume that $h_{\mathrm{top}}(f) < \infty$. Let $\Phi = \{\varphi_n\}$ be an almost additive potential with $P(f, \Phi) > -\infty$ and let ν be a weak Gibbs measure for f with respect to Φ on $\Lambda \subset M$. Assume that either:*

(a) *$\Psi = \{\psi_n\}$ is an asymptotically additive family of potentials, or;*
(b) *$\Psi = \{\psi_n\}$ is a sub-additive family of potentials such that:*
   i. *$\Psi = \{\psi_n\}$ satisfies the weak Bowen condition;*
   ii. *$\inf_{n\geq 1} \frac{\psi_n(x)}{n} > -\infty$ for all $x \in M$; and*
   iii. *the sequence $\{\psi_n/n\}$ is equicontinuous.*

Given $c \in \mathbb{R}$, it holds

$$\overline{R}_\nu(\Psi, [c, \infty)) \leq \sup\big\{ - P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)\big\} \tag{UB}$$

where the supremum is over all $\eta \in \mathcal{M}_f$ such that $\mathcal{F}(\eta, \Psi) \geq c$. Moreover,

$$\underline{R}_\nu(\Psi, (c, \infty)) \geq \sup\big\{ - P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)\big\}$$

where the supremum is taken over all ergodic *measures η satisfying $\mathcal{F}_*(\eta, \Psi) > c$ and $\eta(\Lambda) = 1$. If, in addition, f satisfies specification property then*

$$\underline{R}_\nu(\Psi, (c, \infty)) \geq \sup\big\{ - P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)\big\} \tag{LB}$$

where the supremum is taken over all $\eta \in \mathcal{M}_f$ satisfying $\mathcal{F}_*(\eta, \Psi) > c$ and $\eta(\Lambda) = 1$.

Some comments on our assumptions are in order. We use distinct strategies to deal with the two different classes of potentials Ψ. On the one hand, given a asymptotically additive family of potentials $\Psi = \{\psi_n\}$ the potentials ψ_n can be approximated by Birkhoff sums of continuous potentials and, in particular, the functional $\mu \to \mathcal{F}_*(\mu, \Psi)$ is continuous. On the other hand, for the sub-additive setting conditions (ii) and (iii) will imply the continuity of the previous functional and condition (i) is a bounded distortion property as explained before. In Example 4.1 we explain how to deduce conditions (i) and (iii) in the expanding setting if the family Ψ satisfies some Hölder continuous regularity. Let us also mention that for the lower bound estimate above it is enough the measure μ to satisfy the non-uniform specification property defined in [STV03, Va12], but we shall not use or prove this fact here. Under the uniform hyperbolicity assumption we can also extend a large deviations principle to this non-additive setting as follows.

Corollary A. *Assume that Ω is a basic set for an expanding map f, that $\Phi = \{\varphi_n\}$ is an almost additive sequence of functions such that there exists ν_Φ a Gibbs measure for Φ and $\mu_\Phi \ll \nu_\Phi$ a unique equilibrium state for f with respect to Φ. If $\Psi = \{\psi_n\}$ is a family of potentials as in Theorem B then it satisfies the following large deviations principle: given $c \in \mathbb{R}$ it holds that*

$$\overline{R}_{\nu_\Phi}(\Psi, [c, \infty)) \leq - \inf_{\eta \in \mathcal{M}_f} \{P(f, \Phi) - h_\eta(f) - \mathcal{F}_*(\eta, \Phi) \colon |\mathcal{F}_*(\eta, \Psi) - \mathcal{F}_*(\mu_\Phi, \Psi)| \geq c\}$$

and also

$$\underline{R}_{\nu_\Phi}(\Psi, (c, \infty)) \geq - \inf_{\eta \in \mathcal{M}_f} \{P(f, \Phi) - h_\eta(f) - \mathcal{F}_*(\eta, \Phi) \colon |\mathcal{F}_*(\eta, \Psi) - \mathcal{F}_*(\mu_\Phi, \Psi)| > c\}\,.$$

Let us finish this section with some comments. First notice that the uniqueness of the equilibrium state implies that the later convergence to the average in Kingman's subadditive ergodic theorem is indeed exponential. Finally, since almost additive

potentials are indeed asymptotically additive then our result apply for a wide class of non-additive sequences of observables.

4. Examples and applications

In this section we give applications of our results and discuss specification properties, large deviations results and applications to the study of Lyapunov exponents. Our first example concerns sub-additive families of Hölder continuous potentials.

Example 4.1. *Let X be a compact metric space and assume that $f : X \to X$ expands distances, that is, there are $\lambda > 1$ and $\varepsilon > 0$ such that $d(f(x), f(y)) \geq \lambda\, d(x, y)$ for all $y \in B(x, \varepsilon)$ and, consequently, $f^n : B_n(x, \varepsilon) \to B(f^n(x), \varepsilon)$ is a bijection. Let $\Psi = \{\psi_n\}_n$ be any sub-additive family of γ-Hölder continuous potentials such that the Hölder constants have at most linear growth, meaning that there exists $K > 0$ such that $\text{Höl}_\gamma(\psi_n) \leq Kn$. We claim that Ψ satisfies assumptions (i) and (iii) in Theorem B. On the one hand, $|\psi_n(x) - \psi_n(y)| \leq Kn\, d(x, y)^\gamma \leq Kn\lambda^{-\gamma n} \operatorname{diam}(M)^\gamma$ for every $y \in B_n(x, \varepsilon)$ and $n \geq 1$. In consequence, $\gamma_n(\Phi, \varepsilon) \leq a_n = Kn\lambda^{-\gamma n} \operatorname{diam}(M)^\gamma$ where $\lim_{n\to\infty} a_n/n = 0$. This proves that Ψ satisfies the weak Bowen condition. On the other hand, the sequence $\{\psi_n/n\}$ is Hölder continuous with uniform constant K. Thus this sequence is equicontinuous.*

Now we illustrate an example that do satisfy the almost specification but do not satisfy strong specification property.

Example 4.2. *Consider the piecewise expanding maps of the interval* [0, 1) *given by $T_\beta(x) = \beta x (\bmod 1)$, where $\beta > 1$. This family is known as* beta transformations *and it was introduced by Rényi in* [Ren57]. *It was proved by Buzzi* [Buz97] *that for all but countable many values of β the transformation T_β does not satisfy the specification property. It follows from* [PS07, Th10] *that every β-map satisfies the almost specification property for every unbounded mistake function g. Since the discontinuities have zero entropy our results apply also in the piecewise continuous setting and so for each $\beta > 1$, Theorem A applies to any family of asymptotically additive potentials (e.g. for cocycles as discussed in Example 4.4 over T_β).*

The next example illustrates some applications of our results in the uniformly expanding context.

Example 4.3. *Let $f : M \to M$ be a C^1 map, and let $J \subset M$ be a compact f-invariant set. If J is a maximal topological mixing repeller, Barreira* [Ba06, Page 289] *proved that each almost additive potential $\Phi = \{\varphi_n\}$ with weak Bowen property has a weak Gibbs measure ν_Φ. Thus Theorem B applies to ν_Φ and any asymptotically additive family of potentials Ψ.*

We also point out every almost additive potential indeed satisfies the weak Bowen condition, see [ZZC11, Lemma 2.1] *for a proof. Thus, any almost additive potential has a weak Gibbs measure. Therefore, if J is a maximal topological mixing repeller, Theorem B applies to any almost additive family of potentials Φ and any asymptotically additive family of potentials Ψ without any additional conditions.*

In our next class of examples we estimate the rate of convergence of the maximal Lyapunov exponent for an important class of linear cocycles.

Example 4.4. *Here we consider cocycles over subshifts of finite type considered by Feng and Lau* [FL02] *and later by Feng and Käenmäki* [FK12]*. Let $\sigma : \Sigma \to \Sigma$ be the shift map on the space $\Sigma = \{1, \dots, \ell\}^{\mathbb{N}}$ endowed with the distance $d(x, y) = 2^{-n}$ where $x = (x_j)_j$, $y = (y_j)_j$ and $n = \min\{j \geq 0 : x_j \neq y_j\}$. Given the non-trivial matrices $M_1, \dots, M_\ell \in \mathcal{M}_{d\times d}(\mathbb{C})$, the topological pressure function is defined as*

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\iota\in\Sigma_n} \|M_\iota\|^q$$

where $\Sigma_n = \{1, \dots, d\}^n$ and for any $\iota = (i_1, \dots, i_n) \in \Sigma_n$ one considers the matrix $M_\iota = M_{i_n} \dots M_{i_2} M_{i_1}$. Moreover, for any σ-invariant probability measure μ define also the maximal Lyapunov exponent of μ by

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{\iota\in\Sigma_n} \mu([\iota]) \log \|M_\iota\|$$

and it holds that $P(q) = \sup\{h_\mu(\sigma) + q\, M_*(\mu) : \mu \in \mathcal{M}_\sigma\}$. Notice that this is the variational principle for the potentials $\Psi = \{\psi_n\}$ where $\psi_n(x) = q \log \|M_{\iota_n(x)}\|$ and for any $x \in \Sigma$ we set $\iota_n(x) \in \Sigma_n$ as the only symbol such that x belongs to the cylinder $[\iota_n(x)]$. Assume that the set of matrices $\{M_1, \dots, M_d\}$ is irreducible over $\mathbb{C}^d$, that is, there is no non-trivial subspace $V \subset \mathbb{C}^d$ such that $M_i(V) \subset V$ for all $i = 1, \dots, \ell$. Then it follows from [FK12, Proposition 1.2] *that there exists a unique equilibrium state μ_q for σ with respect to Ψ and it is a Gibbs measure: there exists $C > 0$ such that*

$$\frac{1}{C} \leq \frac{\mu_q([\iota_n])}{e^{-nP(q)} \|M_{\iota_n}\|^q} \leq C$$

for all $\iota_n \in \Sigma_n$ and $n \geq 1$.

Moreover, it is not hard to check that for any $\varepsilon > 0$ we get $y \in B_n(x, \varepsilon)$ if and only if the sequences $x = (x_j)$ and $y = (y_j)_j$ verify $x_j = y_j$ for all $0 \leq j \leq n + [\frac{-\log\varepsilon}{\log 2}]$. In consequence, $B_n(x, \varepsilon) \subset [\iota_n(x)]$ where the potential $\log \|M_{\iota_n(x)}\|$ is constant. Therefore, the sub-additive potentials $\{\log \|M_{\iota_n(x)}\|\}$ clearly satisfy the weak Bowen condition, and Corollary A yields that for any $\delta > 0$

$$\mu_q\Big(x \in \Sigma : \Big|\frac{1}{n} \log \|M_{\iota_n(x)}\| - M_*(\mu)\Big| > \delta\Big)$$

decreases exponentially fast.

The next example combines the theory for both additive and non-additive families of potentials in a non-uniformly expanding context.

Example 4.5. *Let f be the Maneville-Pommeau map on the interval $[0, 1]$ given by $f(x) = x + x^{1+\alpha} (\operatorname{mod} 1)$, for $\alpha \in (0, 1)$. This transformation satisfies the specification property since it is topologically conjugated to the doubling map. Moreover, it is well known that there exists an equilibrium state $\mu \ll Leb$ for f with respect to the potential $\phi = -\log |f'|$ and there exists a tempered sequence K_n such that the measure has the weak Gibbs property:*

$$\frac{1}{K_n} \leq \frac{\mu(\mathcal{P}^{(n)}(x))}{|(f^n)'(x)|} \leq K_n$$

for all $x \in [0, 1]$ and $n \geq 1$, where $\mathcal{P}$ is the Markov partition for f, $\mathcal{P}^{(n)} := \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}$ and $\mathcal{P}^{(n)}(x)$ is the element of the partition $\mathcal{P}^{(n)}$ that contains x. So,

our results apply for any family of asymptotically additive potentials or sub-additive potentials with the weak Gibbs property $\Psi = \{\psi_n\}$.

Let us mention that Yuri [Yu00] *proved that typical piecewise C^1-smooth maps f with indifferent periodic points admit invariant ergodic weak Gibbs measures for $-\log|\det Df|$. In particular our large deviation upper bound results also hold in this context.*

The next class of local diffeomorphisms introduced by Barreira and Gelfert [BG06], satisfying a cone condition, illustrates the existence of sub-additive and sup-additive potentials which are almost additive.

Example 4.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 local diffeomorphism, and let $J \subset \mathbb{R}^2$ be a compact f-invariant set. Following* [BG06]*, we say that f satisfies the following* cone condition *on J if there exist a number $b \le 1$ and for each $x \in J$ there is a one-dimensional subspace $E(x) \subset T_x\mathbb{R}^2$ varying continuous with x such that*

$$Df(x)C_b(x) \subset \{0\} \cup \operatorname{int} C_b(fx)$$

where $C_b(x) = \{(u,v) \in E(x) \bigoplus E(x)^\perp : \ ||v|| \le b||u||\}$. It follows from [BG06, Proposition 4] *that the later condition implies that both families of potentials given by $\{\log\sigma_1(Df^n(x))\}$ and $\{\log\sigma_2(Df^n(x))\}$ are almost additive, where $\sigma_1(L) \ge \sigma_2(L)$ stands for the singular values of the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., the eigenvalues of $(L^*L)^{1/2}$ with L^* denoting the transpose of L. Assume that J is a locally maximal topological mixing repeller of f such that:*

(i) *f satisfies the cone condition on J;*
(ii) *f has tempered distortion on J, i.e., there exists some $\delta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log\sup\Big\{||Df^n(y)(Df^n(z))^{-1}|| : \ x \in J \text{ and } y,z \in B_n(x,\delta)\Big\} = 0,$$

then it follows from [BG06, Theorem 2] *that there exists a weak Gibbs measure ν_{σ_i} with respect to the families of potentials $\{\log\sigma_i(Df^n(x))\}$, for $i = 1,2$. Thus, Theorem B applies to ν_{σ_i} and any family of asymptotically additive potentials or sub-additive potentials satisfying the weak Bowen condition.*

Let us just recall that, given the linear transformations $L = Df(x) : \mathbb{R}^2 \to \mathbb{R}^2$, since $(L^*L)^{1/2}$ is symmetric there exists an orthonormal basis $e_1, \ e_2$ of $\mathbb{R}^2$ which consists of eigenvectors of $(L^*L)^{1/2}$ corresponding respectively to the eigenvalues $\sigma_1(L), \ \sigma_2(L)$. Moreover, the vectors $Le_1, \ Le_2$ are orthogonal and satisfy

$$\sigma_1(L) = ||L|| = ||Le_1|| \text{ and } \sigma_2(L) = ||L^{-1}||^{-1} = ||Le_2||.$$

Therefore, this example gives a sub-additive potential $\{\log\sigma_1(Df^n(x))\}$ and a sup-additive potentials $\{\log\sigma_2(Df^n(x))\}$ which are almost additive.

Finally, in our last example we deal with an example where both families of potentials responsable by computing the largest and smaller Lyapunov exponents are asymptotically additive.

Example 4.7. *Let M be a d-dimensional smooth manifold and J a compact expanding invariant set for a C^1 map f. We say that J is an average conformal repeller if all Lyapunov exponents of each ergodic measure are equal and positive. In particular, it follows from* [BCH10, Theorem 4.2] *that*

$$\lim_{n\to\infty} \frac{1}{n}\left(\log\|Df^n(x)\| - \log\|Df^n(x)^{-1}\|^{-1}\right) = \lim_{n\to\infty} \frac{1}{n} \log\frac{\|Df^n(x)\|}{\|Df^n(x)^{-1}\|^{-1}} = 0$$

uniformly on J. It is easy to see that the family of continuous potentials $\Psi_1 = \{\log \|Df^n(x)\|\}_n$ is sub-additive while $\Psi_2 = \{\log \|Df^n(x)^{-1}\|^{-1}\}_n$ is sup-additive. Furthermore, in this setting it is not hard to check that these two families of potentials are asymptotically additive since they can be uniformly approximated by the additive potentials $\{\frac{1}{d} \log |\det(Df^n(x))|\}_n$. Let Φ be any family of continuous potentials Φ such that f has a unique equilibrium state μ_Φ for f with respect to Φ and that satisfies the Gibbs property. Then it follows from Corollary A that

$$\mu_\Phi\Big(x \in J : \Big|\frac{1}{n}\log \|Df^n(x)\| - \lambda(\mu_\Phi)\Big| > \delta \text{ or } \Big|\frac{1}{n}\log \|Df^n(x)^{-1}\|^{-1} - \lambda(\mu_\Phi)\Big| > \delta\Big)$$

decrease exponentially fast, where in this average conformal setting we consider $\lambda(\mu_\Phi) = \inf_n \frac{1}{n}\int \log \|Df^n(x)\|\, d\mu_\Phi = \sup_n \frac{1}{n}\int -\log \|Df^n(x)^{-1}\|\, d\mu_\Phi$ as the average Lyapunov exponent for μ_Φ. A final remark is that f admits equilibrium states with respect to the potentials Ψ_1 and Ψ_2 by semi-continuity of the metric entropy.

5. Proof of the main results

5.1. **Proof of Proposition A.** We first recall a covering lemma for mistake dynamical balls of points with slow recurrence to the boundary of a partition.

Lemma 5.1. [RVZ12, Lemma 3.2] *Let $\mathcal{Q}$ be a finite partition of M and consider $\varepsilon > 0$ arbitrary small. Let V_ε denote the ε-neighborhood of the boundary $\partial\mathcal{Q}$. For any $\alpha > 0$, there exists $\gamma > 0$(depending only on α), such that for every $x \in M$ satisfying $\sum_{j=0}^{n-1} \chi_{V_\varepsilon}(f^j x) < \gamma n$, the mistake dynamical ball $B_n(g; x, \varepsilon)$ can be covered by $e^{\alpha n}$ cylinders of $\mathcal{Q}^{(n)}$ for sufficiently large n.*

We are now in a position to state and prove our generalized Brin-Katok local entropy formula whose proof exploits the ergodicity of the measure.

Proof of Proposition A. First we note that the limits in the statement of Proposition A are indeed well defined almost everywhere. Given $n \geq 1, \varepsilon > 0$ and $x \in M$ it is clear that $B_n(x, \varepsilon) \subset B_n(g; x, \varepsilon)$. Thus Brin-Katok formula (see [BK83]) immediately yields

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n}\log \mu(B_n(g; x, \varepsilon)) \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n}\log \mu(B_n(x, \varepsilon)) = h_\mu(f)$$

for almost every x. Hence, to complete the proof of the proposition it is enough to show that for μ-almost every x one has

$$\underline{h}_\mu(g; f, x) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n}\log \mu(B_n(g; x, \varepsilon)) \geq h_\mu(f).$$

Fix $\alpha > 0$ arbitrary and let γ be given by Lemma 5.1. Consider a finite partition $\mathcal{Q}$ of M such that $\mu(\partial\mathcal{Q}) = 0$ and $h_\mu(f) \leq h_\mu(f, \mathcal{Q}) + \gamma$. If $\varepsilon > 0$ is small enough the ε-neighborhood V_ε of ∂Q satisfies $\mu(V_\varepsilon) < \gamma/2$. For each positive integer N set

$$\Gamma_N = \Big\{x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \chi_{V_\varepsilon}(f^j(x)) < \gamma \text{ and } \mu(\mathcal{Q}^{(n)}(x)) \leq e^{-n(h_\mu(f,\mathcal{Q})-\gamma)},\ \forall n \geq N\Big\}.$$

Using the ergodicity of μ it follows that $\Gamma_N \subset \Gamma_{N+1}$ and $\mu(\cup_N \Gamma_N) = 1$. By Lemma 5.1 one has that for any $x \in \Gamma_N$ the mistake dynamical ball $B_n(g; x, \varepsilon)$ can

be covered by $e^{\alpha n}$ cylinders of $\mathcal{Q}^{(n)}$. Therefore we obtain that

$$\begin{aligned}\mu(B_n(g;x,\varepsilon)\cap\Gamma_N) &\leq \mu(\cup\{Q\in\mathcal{Q}^{(n)} : B_n(g;x,\varepsilon)\cap Q\neq\emptyset \text{ and } Q\cap\Gamma_N\neq\emptyset\}) \\ &\leq e^{\alpha n}e^{-nh_\mu(f,\mathcal{Q})+\gamma n} \leq e^{\alpha n}e^{-nh_\mu(f)+2\gamma n}\end{aligned}$$

for all $x\in\Gamma_N$ and $n\geq N$. For each $\varepsilon>0$ and positive integer n fixed it follows that $\mu(B_n(g;x,\varepsilon)\cap\Gamma_N)\to\mu(B_n(g;x,\varepsilon))$ as N tends to infinite. Therefore, for any arbitrary constant $\xi>0$ it follows that $\mu(B_n(g;x,\varepsilon))\leq e^{\xi}\mu(B_n(g;x,\varepsilon)\cap\Gamma_N)$ provided that N is large and $x\in\Gamma_N$. In particular, if N is fixed as above then

$$\liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(g;x,\varepsilon)) \geq h_\mu(f)-2\gamma-\alpha-\xi.$$

Considering ε small enough the constants involved above converge to zero proving that $\lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(g;x,\varepsilon))\geq h_\mu(f)$ as claimed. □

5.2. **Proof of Theorem A.** This subsection is devoted to the proof of Theorem A.

Proof of Theorem A (1). Pick $\nu\in\mathcal{E}_f$ with $\mathcal{F}_*(\nu,\Phi)>c$ and let $U_n=\{x\in M : \frac{1}{n}\varphi_n(x)>c\}$. It suffices to show that

$$\liminf_{n\to\infty}\frac{1}{n}\log m(U_n)\geq h_\nu(f)-h_m(g;f,\nu).$$

Without loss of generality, assume $h_m(g;f,\nu)<\infty$, otherwise, there is nothing to prove since $h_\nu(f)\leq h_{top}(f)<\infty$. The strategy for the proof is to approximate the family of asymptotically additive functions by appropriate Birkhoff sums associated to some continuous observable. Indeed, fix a sufficiently small $\xi>0$ so that $\mathcal{F}_*(\nu,\Phi)>c+2\xi$, and φ_ξ is given by (2.1) approximating Φ. We have

$$\widetilde{U}_n=\left\{x\in M : \frac{1}{n}S_n\varphi_\xi(x)>c+\xi\right\}\subset U_n$$

for all sufficiently large n. Moreover, if $\delta>0$ is small enough we can assume that $\mathcal{F}_*(\nu,\Phi)>c+2\xi+\delta$. By uniform continuity of φ_ξ, there exist $\varepsilon_\delta>0$ such that $|\varphi_\xi(x)-\varphi_\xi(y)|<\delta/2$ whenever $d(x,y)\leq\varepsilon_\delta$. We claim that this guarantees that if $\frac{1}{n}S_n\varphi_\xi(x)>c+\xi+\delta$ then $B_n(g;x,\varepsilon)\subset\widetilde{U}_n\subset U_n$ for all sufficiently large n. Indeed, for each $y\in B_n(g;x,\varepsilon)$ there exists $\Lambda\subset I(g;n,\varepsilon_0)$ so that $y\in B_\Lambda(x,\varepsilon)$, and so

$$\sum_{i=0}^{n-1}\varphi_\xi(f^ix)\leq\sum_{i\in\Lambda}[\varphi_\xi(f^iy)+\frac{\delta}{2}]+\sum_{i\notin\Lambda}||\varphi_\xi||\leq\sum_{i=0}^{n-1}[\varphi_\xi(f^iy)+\frac{\delta}{2}]+Cg(n,\varepsilon) \quad (5.1)$$

where $C=2||\varphi_\xi||+\delta$. This implies that $S_n\varphi_\xi(y)\geq S_n\varphi_\xi(x)-Cg(n,\varepsilon)-\delta n/2$. By the definition of mistake function, we have $\frac{1}{n}S_n\varphi_\xi(y)>c+\xi$ for all sufficiently large n, which means that $y\in\widetilde{U}_n\subset U_n$ as claimed.

Fix now an arbitrarily small $\gamma>0$. By the modified Katok entropy formula (2.4) we can choose $0<\varepsilon_1<\varepsilon_\delta$ such that $\liminf_{n\to\infty}\frac{1}{n}\log N(g;n,2\varepsilon,\frac{1}{2})\geq h_\nu(f)-\gamma$ for all $\varepsilon<\varepsilon_1$. Pick also $0<\varepsilon_2<\varepsilon_1$ such that

$$\nu\left(\left\{x\in M : \limsup_{n\to\infty}-\frac{1}{n}\log m(B_n(g;x,\varepsilon_2))\leq h_m(g;f,\nu)+\gamma\right\}\right)>\frac{2}{3}. \quad (5.2)$$

Since $\int\varphi_\xi d\nu>\mathcal{F}_*(\nu,\Phi)-\xi>c+\xi+\delta$, one can use (5.2) and Birkhoff ergodic theorem to choose a measurable set $\mathcal{L}\subset M$ with $\nu(\mathcal{L})\geq\frac{1}{2}$ and a positive integer N such that for any $n\geq N$ the following two properties hold for each $x\in\mathcal{L}$:

(i) $\frac{1}{n} S_n \varphi_\xi(x) > c + \xi + \delta$;

(ii) $m(B_n(g; x, \varepsilon_2)) \geq \exp(-n(h_m(g; f, \nu) + 2\gamma))$.

For each sufficiently large n, let E_n be a maximal set of $(g; n, \varepsilon_2)$-separated points contained in $\mathcal{L}$. It is not hard to check that $\mathcal{L} \subset \bigcup_{x \in E_n} B_n(g; x, 2\varepsilon_2)$ and that the mistake dynamical balls $\{B_n(g; x, \varepsilon_2) : x \in E_n\}$ are disjoint. Using that $\sharp E_n \geq N(g; n, 2\varepsilon_2, \frac{1}{2})$ it follows that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log m(U_n) &\geq \liminf_{n\to\infty} \frac{1}{n} \log m(\widetilde{U}_n) \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in E_n} m(B_n(g; x, \varepsilon_2)) \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \log \sharp E_n \exp(-n(h_m(g; f, \nu) + 2\gamma)) \\
&\geq h_\nu(f) - h_m(g; f, \nu) - 3\gamma.
\end{aligned}$$

The arbitrariness of γ implies the desired result. □

Proof of Theorem A(2). Let $U_n = \{x \in M : \frac{1}{n}\varphi_n(x) \geq c\}$ and $\Psi = \{\psi_n\} \in \mathcal{V}_K^+(g)$. By the definition of $\mathcal{V}_K^+(g)$, there exists a set Υ of full m-measure and constants C_n and ε such that $m(B_n(g; x, \varepsilon)) \leq C_n \exp(-nK + \varphi_n(x))$ for all $x \in \Upsilon$ and $n \geq 1$. We will assume without loss of generality that $U_n \subset \Upsilon$, since otherwise we just consider $U_n \cap \Upsilon$. It suffices to construct a measure $\nu \in \mathcal{M}_f$ with $\mathcal{F}_*(\nu, \Phi) \geq c$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(U_n) \leq -K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi).$$

Let E_n be a maximal $(g; n, \varepsilon)-$separated set contained in U_n. Note that every $(g; n, \varepsilon)$-separated set is also a (n, ε)-separated set. If E_n is not a maximal (n, ε)-separated set consider a maximal (n, ε)-separated set E_n' containing E_n. Now, define the probability measures

$$\mu_n = \sum_{x \in E_n'} \frac{e^{\psi_n(x)}}{Z_n} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$$

where $Z_n = \sum_{x \in E_n'} e^{\psi_n(x)}$ and let ν be a weak* limit of μ_n. It is not hard to check that $\nu \in \mathcal{M}_f$. Moreover, it follows from the proof of [CFH08, Theorem 1.1] that $\limsup_{n\to\infty} \frac{1}{n} \log Z_n \leq h_\nu(f) + \mathcal{F}_*(\nu, \Psi)$. Now, since $U_n \subset \bigcup_{x \in E_n} B_n(g; x, \varepsilon)$ and the constants C_n are tempered we get

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log m(U_n) &\leq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in E_n} m(B_n(g; x, 2\varepsilon)) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in E_n} C_n \exp(-nK + \psi_n(x)) \\
&\leq -K + \limsup_{n\to\infty} \frac{1}{n} \log Z_n \\
&\leq -K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi).
\end{aligned}$$

It remains to prove that $\mathcal{F}_*(\nu, \Phi) \geq c$. For each small $\xi > 0$, let φ_ξ be given by (2.1) approximating Φ. Since μ_n is a linear combination of measures $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}$

for $x \in E'_n$ and

$$\int \varphi_\xi \,\mathrm{d}\Big(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x}\Big) = \frac{1}{n}S_n\varphi_\xi(x) \ge \frac{1}{n}\varphi_n(x) - \xi \ge c - \xi,$$

we deduce that $\int \varphi_\xi \, d\mu_n \ge c - \xi$. In consequence $c - \xi \le \int \varphi_\xi \mathrm{d}\nu \le \mathcal{F}_*(\nu,\Phi) + \xi$. The arbitrariness of ξ implies that $\mathcal{F}_*(\nu,\Phi) \ge c$. This completes the proof of the second part of the theorem. □

Proof of Theorem A (3). Let $U_n = \{x \in M : \frac{1}{n}\varphi_n(x) \ge c\}$ and $\Psi = \{\psi_n\} \in \mathcal{V}_K^-(g)$. Without loss of generality, assume that $U_n \subset \Upsilon$, since otherwise we just consider $U_n \cap \Upsilon$. Pick an arbitrary f-invariant and ergodic measure ν satisfying that $\mathcal{F}_*(\nu,\Phi) > c$ and $\nu(\Upsilon) = 1$. It follows from the definition of $\mathcal{V}_K^-(g)$ and the ergodicity of ν that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log m(B_n(g;x,\varepsilon)) \le K - \mathcal{F}_*(\nu,\Psi), \quad \nu - a.e.\ x \in \Upsilon,$$

and, consequently, $h_m(g; f,\nu) \le K - \mathcal{F}_*(\nu,\Psi)$. Therefore, it follows from the first item of this theorem that $\underline{R}_m(\Phi,(c,\infty)) \ge \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu,\Psi)\}$, where the supremum is taken over all $\nu \in \mathcal{E}_f$ satisfying that $\mathcal{F}_*(\nu,\Phi) > c$ and $\nu(\Upsilon) = 1$. This proves the third assertion of the theorem. □

Proof of Theorem A(4). Fix an asymptotically additive family of potentials $\Psi \in \mathcal{V}_K^-$. Let $U_n = \{x \in M : \frac{1}{n}\varphi_n(x) > c\}$ and assume, without loss of generality, that $U_n \subset \Upsilon$. Pick an invariant measure $\nu \in \mathcal{M}_f$ with $\mathcal{F}_*(\nu,\Phi) > c$ and $\nu(\Upsilon) = 1$. It suffices to prove that

$$\liminf_{n\to\infty}\frac{1}{n}\log m(U_n) \ge -K + h_\nu(f) + \mathcal{F}_*(\nu,\Psi) - 5\gamma$$

for any preassigned $\gamma > 0$. We will divide the proof in several lemmas.

Lemma 5.2. *For every $\delta > 0$ and $\gamma > 0$, there exists $\mu \in \mathcal{M}_f$ such that $\mu = \sum_{i=1}^k a_i\mu_i$, where $\sum_{i=1}^k a_i = 1$ and $\mu_i \in \mathcal{E}_f$, satisfying that*

(i) $\mu_i(\Upsilon) = 1$, $i = 1,\cdots,k$;
(ii) $|\mathcal{F}_*(\nu,\Phi) - \mathcal{F}_*(\mu,\Phi)| < \delta$;
(iii) $h_\mu(f) + \mathcal{F}_*(\mu,\Psi) \ge h_\nu(f) + \mathcal{F}_*(\nu,\Psi) - 2\gamma$.

Proof. Although the argument follows by a small modification of the standard one in [You90] we will prove it here for completeness. Let dist^* denote the metric on the space of all Borel probability measures on M. It follows by uniform continuity of the functional $\eta \mapsto \mathcal{F}_*(\eta,\Phi)$ (see [FH10]) that for any given $\delta > 0$ and $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that

$$\mathrm{dist}^*(\tau_1,\tau_2) < \varepsilon_0 \Rightarrow |\mathcal{F}_*(\tau_1,\Phi) - \mathcal{F}_*(\tau_2,\Phi)| < \delta \text{ and } |\mathcal{F}_*(\tau_1,\Psi) - \mathcal{F}_*(\tau_2,\Psi)| < \gamma.$$

Let $\mathcal{P} = \{P_1,\cdots,P_2\}$ be a partition of $\mathcal{M}_f$ with $\mathrm{diam}\mathcal{P} = \max_{1\le i\le k}|P_i| < \varepsilon_0$, where $|\cdot|$ denotes the diameter of a set. Using ergodic decomposition theorem, there is a Borel probability measure π on $\mathcal{M}_f$ such that $\pi(\mathcal{E}_f) = 1$ and $\mathcal{F}_*(\nu,\mathcal{G}) = \int \mathcal{F}_*(\tau,\mathcal{G})\mathrm{d}\pi(\tau)$ for any asymptotically additive potential $\mathcal{G}$. We refer the reader to [FH10] for a proof of this fact. Moreover, let $\mathcal{E}'_f = \{\tau \in \mathcal{E}_f :\ \tau(\Upsilon) = 1\}$ then

$\pi(\mathcal{E}'_f) = 1$. Let $a_i = \pi(P_i)$ and pick $\mu_i \in P_i \cap \mathcal{E}'_f$ satisfying $h_{\mu_i}(f) > h_\tau(f) - \gamma$ for all $\tau \in P_i \cap \mathcal{E}'_f$. Then we have that

$$a_i h_{\mu_i}(f) \geq \int_{P_i \cap \mathcal{E}'_f} [h_\tau(f) - \gamma] \, \mathrm{d}\pi(\tau) = \int_{P_i} h_\tau(f) \mathrm{d}\pi(\tau) - \gamma a_i$$

Summing over i it follows that the probability measure $\mu = \sum_{i=1}^k a_i \mu_i$ satisfies $h_\mu(f) \geq h_\nu(f) - \gamma$. Moreover,

$$\mathcal{F}_*(\mu, \Psi) = \sum_{i=1}^k a_i \mathcal{F}_*(\mu_i, \Psi) \geq \sum_{i=1}^k \int_{P_i} (\mathcal{F}_*(\tau, \Psi) - \gamma) \mathrm{d}\pi(\tau) = \mathcal{F}_*(\nu, \Psi) - \gamma.$$

Hence, we have that $h_\mu(f) + \mathcal{F}_*(\mu, \Psi) \geq h_\nu(f) + \mathcal{F}_*(\nu, \Psi) - 2\gamma$. A similar argument shows that $|\mathcal{F}_*(\nu, \Phi) - \mathcal{F}_*(\mu, \Phi)| < \delta$. This completes the proof of the lemma. $\square$

Fix a small $\xi > 0$, let ψ_ξ and φ_ξ be the continuous functions given as in (2.1) approximating the sequences Ψ and Φ respectively. Fix $\delta = (\mathcal{F}_*(\nu, \Phi) - c)/4 > 0$ and a small $\gamma > 0$. Using the uniformly continuity of ψ_ξ and φ_ξ, Birkhoff ergodic theorem and Proposition A we can choose $\varepsilon > 0$ sufficiently small and N sufficiently large so that the following properties hold:

(i) $d(x, y) < \varepsilon \Rightarrow |\psi_\xi(x) - \psi_\xi(y)| < \gamma$ and $|\varphi_\xi(x) - \varphi_\xi(y)| < \delta$;

(ii) For each $n \geq N$ and $1 \leq i \leq k$, there exist at least $e^{[a_i n](h_{\mu_i}(f) - \gamma)}$ points $x_1^{(i)}, \cdots, x_{n_i}^{(i)}$ that are $(2g; [a_i n], 4\varepsilon)$-separated with the property that: for each $1 \leq j \leq n_i$, we have
  - (a) $S_{[a_i n]} \psi_\xi(x_j^{(i)}) \leq [a_i n](\int \psi_\xi \mathrm{d}\mu_i + \gamma)$;
  - (b) $S_{[a_i n]} \varphi_\xi(x_j^{(i)}) \geq [a_i n](\int \varphi_\xi \mathrm{d}\mu_i - 2\delta)$.

Note that $n_i \geq e^{[a_i n](h_{\mu_i}(f) - \gamma)}$. Since f has the $g-$almost specification property, for each $k-$tuple $(j_1, \cdots, j_k)$ with $1 \leq j_i \leq n_i$ we can choose a point

$$y = y_{j_1 \cdots j_k} \in \bigcap_{i=1}^k f^{-\sum_{j=0}^{i-1} [a_i n]}(B_{[a_i n]}(g; x_{j_i}, \varepsilon)).$$

Let $F = \{y_{j_1 \cdots j_k} : (j_1, \cdots, j_k), \, 1 \leq j_i \leq n_i\}$ and $\hat{n} = \sum_i [a_i n]$. Now we prove:

Lemma 5.3. *The set F is $(\hat{n}, 2\varepsilon)$-separated, i.e., $B_{\hat{n}}(y, \varepsilon) \cap B_{\hat{n}}(y', \varepsilon) = \emptyset$ for any disctinct $y, y' \in F$. In particular $\# F \geq \exp(\hat{n}(h_\mu(f) - \gamma))$.*

Proof. Given $y \neq y' \in F$, take distinct k-tuples $(j_1, \cdots, j_k) \neq (j'_1, \cdots, j'_k)$ such that $y = y_{j_1 \cdots j_k}$ and $y' = y_{j'_1 \cdots j'_k}$. We assume without loss of generality that $j_1 \neq j'_1$. Consequently, $y \in B_{[a_1 n]}(g; x_{j_1}, \varepsilon)$ and $y' \in B_{[a_1 n]}(g; x_{j'_1}, \varepsilon)$. Therefore, there exists $\Lambda_i \in I(g; [a_1 n], \varepsilon)$ for $i = 1, 2$ such that $d_{\Lambda_1}(x_{j_1}, y) < \varepsilon$ and $d_{\Lambda_2}(x_{j'_1}, y') < \varepsilon$. If $\Lambda = \Lambda_1 \cap \Lambda_2$ then $[a_1 n] \geq \# \Lambda \geq [a_1 n] - 2g([a_1 n], \varepsilon])$ and so

$$\begin{aligned} d_{\hat{n}}(y, y') &\geq d_\Lambda(y, y') \geq d_\Lambda(x_{j_1}, x_{j'_1}) - d_\Lambda(y, x_{j_1}) - d_\Lambda(y', x'_{j_1}) \\ &> 4\varepsilon - \varepsilon - \varepsilon = 2\varepsilon. \end{aligned}$$

Hence $B_{\hat{n}}(y, \varepsilon) \cap B_{\hat{n}}(y', \varepsilon) = \emptyset$ as claimed. By the previous reasoning all $y = y_{j_1 \cdots j_k}$ are distinct and so the cardinality of F is bounded from below by $n_1 \cdots n_k \geq \exp(\hat{n}(h_\mu(f) - \gamma))$. This finishes the proof of the lemma. $\square$

We proceed with the following auxiliary lemma.

Lemma 5.4. *If $\xi, \delta > 0$ are small enough then for every large n and $y \in F$:*

(i) $B_{\hat{n}}(y,\varepsilon) \subset U_{\hat{n}}$;
(ii) $\frac{1}{\hat{n}}\psi_{\hat{n}}(y) \geq \mathcal{F}_*(\mu,\Psi) - 2\gamma$.

Proof. Fix a small $\xi > 0$ and let φ_ξ be a continuous function approximating the sequence Φ as before. For each $z \in B_{\hat{n}}(y,\varepsilon)$ with $y = y_{j_1\cdots j_k}$ consider the images $y_i = f^{([a_1 n]+\dots+[a_{i-1}n])}(y)$ $(1 \leq i \leq k)$ and notice that

$$\begin{aligned}
\frac{1}{\hat{n}}\varphi_{\hat{n}}(z) &\geq \frac{1}{\hat{n}}S_{\hat{n}}\varphi_\xi(z) - \xi \geq \frac{1}{\hat{n}}\sum_{i=1}^{k}\big(S_{[a_i n]}\varphi_\xi(y_i) - [a_i n]\delta\big) - \xi \\
&\geq \frac{1}{\hat{n}}\sum_{i=1}^{k}\Big[S_{[a_i n]}\varphi_\xi(x_{j_i}) - 2[a_i n]\delta - 2g([a_i n],\varepsilon)||\varphi_\xi||\Big] - \xi \\
&\geq \frac{1}{\hat{n}}\sum_{i=1}^{k}\Big[[a_i n]\Big(\int\varphi_\xi\,\mathrm{d}\mu_i - 2\delta\Big) - 2g([a_i n],\varepsilon)||\varphi_\xi||\Big] - 2\delta - \xi \\
&\geq \frac{1}{\hat{n}}\sum_{i=1}^{k}\Big[[a_i n]\mathcal{F}_*(\mu_i,\Phi) - 2g([a_i n],\varepsilon)||\varphi_\xi||\Big] - 4\delta - 2\xi \\
&\geq \mathcal{F}_*(\nu,\Phi) - 5\delta - 2\xi
\end{aligned}$$

for all sufficiently large n. Therefore, $\frac{1}{\hat{n}}\varphi_{\hat{n}}(z) > c$ provided that δ and ξ are sufficiently small. This completes the proof of (i) above. Since the arguments to prove (ii) are analogous we shall omit the proof. $\square$

We are now in a position to finish the proof of Theorem A(4). Indeed, combining the previous lemmas we obtain that

$$\begin{aligned}
\liminf_{\hat{n}\to\infty}\frac{1}{\hat{n}}\log m(U_{\hat{n}}) &\geq \liminf_{\hat{n}\to\infty}\frac{1}{\hat{n}}\log\sum_{y\in F} m(B_{\hat{n}}(y,\varepsilon)) \\
&\geq \liminf_{\hat{n}\to\infty}\frac{1}{\hat{n}}\log\sum_{y\in F} C_{\hat{n}}\exp(-\hat{n}K + \psi_{\hat{n}}(y)) \\
&\geq -K + h_\mu(f) + \mathcal{F}_*(\mu,\Psi) - 3\gamma \\
&\geq -K + h_\nu(f) + \mathcal{F}_*(\nu,\Psi) - 5\gamma,
\end{aligned}$$

which proves the last statement and finishes the proof of the theorem. $\square$

5.3. **Proof of Theorem B.** Here we prove our second main result, that estimates the measure of the deviation sets in terms of some thermodynamical quantities for asymptotically additive and sub-additive families of potentials.

Case I. $\Psi = \{\psi_n\}_n$ *asymptotically additive sequence of continuous functions*

Upper bound. Since the upper bound is similar for both asymptotically additive and sub-additive potentials we shall focus on the first case. So, let $\Psi = \{\psi_n\} \in \mathcal{A}$ be an asymptotically additive sequence of continuous functions, ν be a weak Gibbs measure with respect to Φ on $\Lambda \subset M$ and take $c \in \mathbb{R}$. Therefore, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a sequence of positive constants $(K_n)_{n\geq 1}$ (depending only on ε) satisfying $\lim_{n\to\infty}\frac{1}{n}\log K_n = 0$ and

$$K_n^{-1} \leq \frac{\nu(B_n(x,\varepsilon))}{e^{-nP(f,\Phi)+\varphi_n(x)}} \leq K_n,\ \forall n \geq 1 \text{ and } x \in \Lambda.$$

In consequence $\Phi \in \mathcal{V}_K^+(g)$ with respect to the reference measure ν, where we consider the mistake function $g \equiv 0$, $C_n = K_n$ and $K = P(f, \Phi)$. In this way, (UB) is a direct consequence of Theorem A (2). This finishes the proof of the upper bound in Theorem B.

Lower bound using ergodic measures. The same reasoning as above yields that $\Phi \in \mathcal{V}_K^-(g)$ with respect to the reference measure ν, where we consider the mistake function $g \equiv 0$, $C_n = K_n^{-1}$ and $K = P(f, \Phi)$. In this way, the lower bound estimate using ergodic measures is a direct consequence of Theorem A (3). However, for completeness we shall prove this fact to collect some constants needed to the proof of (LB). Let $\Psi = \{\psi_n\}$ be an asymptotically additive sequence of continuous functions, $c \in \mathbb{R}$ be fixed and $\beta > 0$ be arbitrary. Denote by U_n the set of points $x \in M$ so that $\psi_n(x) > cn$, without loss of generality, we assume that $U_n \subset \Lambda$. We claim that if η is any f-invariant and ergodic probability measure so that $\eta(\Lambda) = 1$ and $\mathcal{F}_*(\eta, \Psi) > c$ then

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu(U_n) \ge -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi) - 2\beta$$

for any preassigned $\beta > 0$ with $\mathcal{F}_*(\eta, \Psi) > c + \beta$. By ergodicity one can pick $N \ge 1$ large and ε_0 small so that $N\left(n, 2\varepsilon, \frac{1}{2}\right) \ge e^{[h_\eta(f)-\beta]n}$ for all $0 < \varepsilon < \varepsilon_0$ and $n \ge N$, and also the set F of points $x \in M$ satisfying $\mathcal{F}_*(\eta, \Phi) - \beta < \frac{1}{n}\varphi_n(x) < \mathcal{F}_*(\eta, \Phi) + \beta$ and $\mathcal{F}_*(\eta, \Psi) - \beta < \frac{1}{n}\psi_n(x) < \mathcal{F}_*(\eta, \Psi) + \beta$ for all $n \ge N$ has η-measure at least $\frac{1}{2}$.

Now, take $\xi = (\mathcal{F}_*(\eta, \Psi) - \beta - c)/4 > 0$ and let ψ_ξ be given by approximation (2.1). Up to consider smaller ε_0 and larger $N \ge 1$ we may assume that $|\psi_\xi(x) - \psi_\xi(y)| < \xi$ whenever $d(x, y) < \varepsilon_0$ (ψ_ξ is uniformly continuous) and $\|\psi_n - S_n\psi_\xi\| \le \xi n$ for all $n \ge N$. Using our construction it is clear that $F \subset U_n$. We claim that

$$B_n(x, \varepsilon) \subset U_n \quad \text{for all } x \in F. \tag{5.3}$$

In fact, if $x \in F$ and $y \in B_n(x, \varepsilon)$ using $\frac{1}{n}\psi_n(x) \ge \mathcal{F}_*(\eta, \Psi) - \beta = c + 4\xi$ we get

$$|\psi_n(x) - \psi_n(y)| \le |\psi_n(x) - S_n\psi_\xi(x)| + |\psi_n(y) - S_n\psi_\xi(y)| + |S_n\psi_\xi(x) - S_n\psi_\xi(y)| < 3\xi n$$

and consequently $\psi_n(y) > cn$, proving our claim. Therefore $U_n \supset \bigcup_{x\in F} B_n(x, \varepsilon) \supset F$ for every $0 < \varepsilon < \varepsilon_0$ and $n \ge N$. Moreover, if $F_n \subset F$ is a maximal $(n, 2\varepsilon)$-separated set, one can use that the dynamical balls $B_n(x, \varepsilon)$ centered at points in F_n are pairwise disjoint and contained in U_n. This yields

$$\begin{aligned}
\nu(U_n) &\ge \nu\Big(\bigcup_{x\in F_n} B_n(x, \varepsilon)\Big) = \sum_{x\in F_n} \nu\Big(B_n(x, \varepsilon)\Big) \\
&\ge \sum_{x\in F_n} K_n^{-1} e^{-P(f,\Phi)n + \varphi_n(x)} \\
&\ge K_n^{-1} \exp(-P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi) - 2\beta)n
\end{aligned}$$

whenever $n \ge N$, which proves our claim since the sequence $\{K_n\}$ is tempered. The arbitrariness of β and the measure implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\Big\{x \in M : \frac{1}{n}\psi_n(x) > c\Big\} \ge -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Psi)$$

for every f-invariant, ergodic probability measure η such that $\mathcal{F}_*(\eta, \Psi) > c$ and $\eta(\Lambda) = 1$. In consequence we obtain the second assertion in Theorem B.

Lower bound over all invariant measures. Here we deduce the general lower bound over all invariant measures under the assumption that f satisfies some weak specification properties.

Fix $c > 0$ and an asymptotically additive potential $\Psi = \{\psi_n\}$. Let $U_n = \{x \in M : \frac{1}{n}\psi_n(x) > c\}$. Without loss of generality, we assume that $U_n \subset \Lambda$. We claim that for any f-invariant probability measure η satisfying $\eta(\Lambda) = 1$ and $\mathcal{F}_*(\eta, \Psi) > c$ it holds that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left[x \in M : \frac{1}{n}\psi_n(x) > c \right] \geq -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi).$$

We will make use of the following approximation result that is a reformulation of Lemma 5.2 and in a position to finish the proof of the theorem for asymptotically additive sequences.

Lemma 5.5. *Let $\eta = \int \eta_x d\eta(x)$ be the ergodic decomposition of an f-invariant probability measure η such that $\eta(\Lambda) = 1$. Given $\gamma > 0$ and a finite set of asymptotically additive potentials $(\Psi_j)_{1\leq j\leq r}$, there are positive real numbers $(a_i)_{1\leq i\leq k}$ satisfying $a_i \leq 1$ and $\sum a_i = 1$, and finitely many points $x_1, \ldots, x_k$ such that the ergodic measures $\eta_i = \eta_{x_i}$ from the ergodic decomposition satisfy*

(i) $\eta_i(\Lambda) = 1$*;*
(ii) $h_{\hat\eta}(f) \geq h_\eta(f) - \gamma$*; and*
(iii) $|\mathcal{F}_*(\Psi_j, \hat\eta) - \mathcal{F}_*(\Psi_j, \eta)| < \gamma$ *for every* $1 \leq j \leq r$*;*

where $\hat\eta = \sum_{i=1}^{k} a_i \eta_i$.

Given a small $\gamma > 0$, using the lemma, there are ergodic probability measures $(\eta_i)_i$ so that a probability measure $\hat\eta = \sum_{i=1}^k a_i\eta_i$ satisfies $h_{\hat\eta}(f) \geq h_\eta(f) - \gamma$, and such that $|\mathcal{F}_*(\Psi, \hat\eta) - \mathcal{F}_*(\Psi, \eta)| < \gamma$ and $|\mathcal{F}_*(\Phi, \hat\eta) - \mathcal{F}_*(\Phi, \eta)| < \gamma$. Proceeding as before there are $N \geq 1$ large and ε_0 small so that for every $1 \leq i \leq k$, the set F^i of points $x \in \Lambda$ such that $\mathcal{F}_*(\eta_i, \Phi) - \gamma < \frac{1}{n}\varphi_n(x) < \mathcal{F}_*(\eta_i, \Phi) + \gamma$, and $\mathcal{F}_*(\eta_i, \Psi) - \gamma < \frac{1}{n}\psi_n(x) < \mathcal{F}_*(\eta_i, \Psi) + \gamma$ for every $n \geq N$ and $0 < \varepsilon < \varepsilon_0$ has η_i-measure at least $\frac{1}{2}$. We may assume also without loss of generality that the minimum number of $(n, 4\varepsilon)-$dynamical balls needed to cover a set of η_i-measure $\frac{1}{2}$ satisfies $N_i\left(n, 4\varepsilon, \frac{1}{2}\right) \geq e^{[h_{\eta_i}(f)-\gamma]n}$ for all $0 < \varepsilon < \varepsilon_0$ and $n \geq N$. So, pick a finite set $F_n^i \subset F^i$ so that F_n^i is a maximal $([a_i n], 4\varepsilon)$-separated set in F^i and $\#F_n^i \geq e^{(h_{\eta_i}(f)-\gamma)\,[a_i n]}$. Moreover, by the specification property, for every sequence $(x_1, x_2, \ldots, x_k)$ with $x_i \in F_n^i$ there exists $x \in M$ that ε-shadows each x_i during $[a_i n]$ iterates with a time lag of $N(\varepsilon, n)$ iterates in between. Set $\tilde n = \sum_i [a_i n] + kN(\varepsilon, n)$ and consider the set

$$\mathcal{L}_{\tilde n} = \bigcup \{B_{\tilde n}(x, \varepsilon) : x = x_{i_1,\ldots,i_k}\}.$$

Similar arguments as in the proof of Lemma 5.3 and Lemma 5.4, we can show that the dynamical balls in the later union have the following properties:

i. the dynamical balls $B_{\tilde n}(x, \varepsilon)$ and $B_{\tilde n}(y, \varepsilon)$ are disjoint for $x \neq y$;
ii. the number of the dynamical balls is larger than $\exp \tilde n(h_{\hat\eta}(f) - \gamma)$;
iii. each dynamical ball $B_{\tilde n}(x, \varepsilon)$ is contained in $U_{\tilde n}$;
iv. $\frac{1}{\tilde n}\varphi_{\tilde n}(y) \geq \mathcal{F}_*(\hat\eta, \Phi) - 2\gamma$ for every $y \in \mathcal{L}_{\tilde n}$.

Since ν is a weak Gibbs measure for Φ on $\Lambda \subset M$ and $\tilde{n} \geq N$ it follows that $\nu(B_{\tilde{n}}(x,\varepsilon)) \geq K_{\tilde{n}}^{-1} e^{-\tilde{n}P(f,\Phi)+\varphi_{\tilde{n}}(x)}$. Therefore, it yields that

$$\begin{aligned}\nu(U_{\tilde{n}}) &\geq \sum_x \nu(B_{\tilde{n}}(x,\varepsilon)) \geq K_{\tilde{n}}^{-1} \exp[-P(f,\Phi) + h_{\hat{\eta}}(f) + \mathcal{F}_*(\hat{\eta},\Phi) - 3\gamma]\tilde{n} \\ &\geq K_{\tilde{n}}^{-1} \exp[-P(f,\Phi) + h_{\eta}(f) + \mathcal{F}_*(\eta,\Phi) - 5\gamma]\tilde{n}\end{aligned}$$

for every large n. Since the sequence $\{K_n\}$ is tempered and γ was chosen arbitrarily then we get

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu(U_n) \geq -P(f,\Phi) + h_\eta(f) + \mathcal{F}_*(\eta,\Phi)$$

which finishes the proof of Theorem B in the case of asymptotically additive sequences Ψ.

Case II. $\Psi = \{\psi_n\}_n$ *sub-additive sequence of continuous potentials*

Through the remaining of the proof, assume that $\Psi = \{\psi_n\}_n$ is a sub-additive sequence of continuous potentials that satisfy the weak Bowen condition, that $(\frac{\psi_n}{n})_n$ is equicontinuous and that $\inf_{n\geq 1} \frac{\psi_n(x)}{n} > -\infty$ for every $x \in M$ as in Theorem B. Since the proof of upper and lower bounds are similar to the previous ones for asymptotically additive potentials we will only highlight the main differences.

Upper bound. Let $\Psi = \{\psi_n\} \in \mathcal{A}$ be as above, ν be a weak Gibbs measure with respect to Φ on $\Lambda \subset M$ and take $c \in \mathbb{R}$. Therefore, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a sequence of positive constants $(K_n)_{n\geq 1}$ (depending only on ε) satisfying $\lim\limits_{n\to\infty} \frac{1}{n} \log K_n = 0$ and

$$K_n^{-1} \leq \frac{\nu(B_n(x,\varepsilon))}{e^{-nP(f,\Phi)+\varphi_n(x)}} \leq K_n, \ \forall n \geq 1 \text{ and } x \in \Lambda.$$

In consequence $\Phi \in \mathcal{V}_K^+(g)$ with respect to the reference measure ν, where we consider the mistake function $g \equiv 0$, $C_n = K_n$ and $K = P(f,\Phi)$. On the other hand, using that $\{\frac{\psi_n}{n}\}$ is equicontinuous, $\psi_n(x) \leq n\|\psi_1\|$ and that $\inf_{n\geq 1} \frac{\psi_n(x)}{n} > -\infty$ for every $x \in M$ it follows from Arzéla-Ascoli theorem that the sequence $\{\frac{\psi_n}{n}\}$ admits a subsequence uniformly convergent to some continuous function g. Therefore, for any $\varepsilon > 0$ small there exists a positive integer $k \geq 1$ such that $\|\frac{\psi_k}{k} - g\| < \varepsilon$. For each sufficiently large n, it follows from [CHZ, Lemma 2.2] that there exists a constant C depending only on k such that $\psi_n \leq S_n(\frac{\psi_k}{k}) + C$. Consequently, we have that

$$\frac{\psi_n(x)}{n} \leq \frac{1}{n} S_n\Big(\frac{\psi_k}{k}\Big)(x) + \frac{C}{n} \leq \frac{1}{n} S_n g(x) + 2\varepsilon$$

for all x and sufficiently large n. Thus, for any $\varepsilon > 0$ one has the inclusion

$$\Big\{\frac{1}{n}\psi_n \geq c\Big\} \subset \Big\{\frac{1}{n} S_n g \geq c - 2\varepsilon\Big\}$$

provided that n is large. Now we proceed as in the proof of Theorem A (2). Fix $\varepsilon > 0$ be arbitrary small and take a maximal (n,ε)-separated set E_n contained in $\{S_n g \geq (c-2\varepsilon)n\}$. Then, the same arguments as before show that any weak* limit

μ of the probability measures

$$\mu_n = \sum_{x \in E_n} \frac{e^{\varphi_n(x)}}{Z_n} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$$

where $Z_n = \sum_{x \in E_n} e^{\varphi_n(x)}$, satisfies (UB).

It remains to prove that $\mathcal{F}_*(\mu, \Psi) \geq c$. Applying sub-additive ergodic theorem to the invariant measure μ, there exist a f-invariant function $\tilde{\psi}$ and a subset Υ with $\mu(\Upsilon) = 1$ such that

$$\tilde{\psi}(x) = \lim_{n \to \infty} \frac{\psi_n(x)}{n}, \ \forall x \in \Upsilon.$$

This implies that $\tilde{\psi}(x) = g(x)$ for each point $x \in \Upsilon$, and consequently,

$$\mathcal{F}_*(\mu, \Psi) = \int \tilde{\psi} \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu.$$

Since the same argument holds for any f-invariant probability measure this also implies that the map $\mu \mapsto \mathcal{F}_*(\mu, \Psi)$ is continuous since the function g is continuous. Proceed as in the proof of theorem A(2), we have

$$\int g \, \mathrm{d}\mu = \lim_{n \to \infty} \int g \, \mathrm{d}\mu_n \geq c - 2\varepsilon.$$

The arbitrariness of ε implies that $\mathcal{F}_*(\mu, \Psi) \geq c$. This finishes the proof of the upper bound in this sub-additive setting.

Lower bounds. The proof of this second case goes along the same lines of the first one. For that reason we shall just highlight the differences. Fix $c > 0$ and let $\Psi = \{\psi_n\}$ be a sub-additive family of continuous potentials as above, and assume without loss that $U_n = \{x \in M : \psi_n(x) > cn\} \subset \Lambda$. We claim that

$$\liminf_{n \to \infty} \frac{1}{n} \log \nu \left[x \in M : \frac{1}{n} \psi_n(x) > c \right] \geq -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi).$$

for any f-invariant probability measure η so that $\eta(\Lambda) = 1$ and $\mathcal{F}_*(\eta, \Psi) > c$. In fact, the need of the former asymptotically additive assumption used to deduce the claim for ergodic measures was only to control the variation of this sequence on dynamical balls. See (5.3) for the precise statement. To overlap this difficulty in this setting we use the weak Bowen property as follows.

Lemma 5.6. *Let $\Psi = \{\psi_n\}_n$ be a sub-additive potentials with the weak Bowen property and $a \in \mathbb{R}$. If $\psi_n(x) > an$ then for all $\gamma > 0$ there exists $N \geq 1$ large so that all $y \in B_n(x, \varepsilon)$ satisfy $\psi_n(y) > (a - \gamma)n$*

Proof. Let $a \in \mathbb{R}$ and $\gamma > 0$ be arbitrary. In fact, using that $\mathrm{var}(\psi_n \mid_{B_n(x,\varepsilon)}) \leq a_n$ for some sequence $a_n/n \to 0$ it is immediate that $|\psi_n(x) - \psi_n(y)| \leq a_n$ and consequently

$$\frac{1}{n} \psi_n(y) \geq \frac{1}{n} \psi_n(x) - \frac{a_n}{n} \geq a - \gamma$$

for every $y \in B_n(x, \varepsilon)$, provided that n is large. This proves the lemma. $\square$

Hence the remaining of the argument to prove the claim in the case that η is ergodic is analogous to the one of Case I. In the case that η is not ergodic an extra approximation argument by ergodic measures given by Lemma 5.5 is necessary. The proof of that lemma uses the continuity of functional $\mu \to \mathcal{F}_*(\Psi, \mu)$. This is

not necessarily continuous for general sub-additive sequences but in our context $\mathcal{F}_*(\Psi, \mu) = \int g\, d\mu$ varies continuously with μ. Since the remaining argument in the proof follows the one in Case I we shall omit the details. This finishes the proof of Theorem B.

Appendix A: Estimates for measures of mistake dynamical balls

In this appendix we provide an estimate for the measure of mistake dynamical balls with respect to the Gibbs measures obtained through additive thermodynamical formalism for uniformly expanding transformations. In this setting there is a unique equilibrium state for f with respect to any Hölder continuous potential ϕ, and it is equivalent to the unique Gibbs measure for f with respect to ϕ.

Proposition 5.1. *Let X be a compact manifold, $f : X \to X$ be a uniformly expanding map, ϕ be a Hölder continuous potential and $\mu = \mu_\phi$ be the unique equilibrium state for f with respect to ϕ. Let g be any mistake function. There exists $C > 0$ and for any $\xi > 0$ there exists a measurable set $X_\xi \subset X$ such that $\mu(X_\xi) \geq 1 - \xi$ and*

$$e^{-Pn+S_n\phi(x)} \leq \mu(B_n(x,\varepsilon)) \leq \mu(B_n(g;x,\varepsilon)) \leq Ce^{2\xi n-Pn+S_n\phi(x)}$$

for all $x \in X_\xi$, $n \geq 1$ and small ε, where $P = P_{top}(f,\phi)$ denotes the topological pressure of f with respect to Φ.

Proof. Let $\xi > 0$ be arbitrary and fixed. Since f is uniformly expanding then there exists a finite Markov partition $\mathcal{Q}$. Moreover, the unique equilibrium state μ for f with respect to ϕ verifies $\mu(\partial\mathcal{Q}) = 0$ and satisfies the Gibbs property: there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{\mu(\mathcal{Q}^n(x))}{e^{-Pn+S_n\phi(x)}} \leq C$$

for all $x \in X$ and every $n \neq 1$, where P is the topological pressure of f with respect to ϕ and $\mathcal{Q}^n(x))$ is the element of the partition $\mathcal{Q}^n = \vee_{j=0}^{n-1} f^{-j}(\mathcal{Q})$ that contains x. Using Lemma 5.1, it follows that there exists a measurable set $X_\xi \subset X$ such that $\mu(X_\xi) \geq 1 - \xi$ and for which $B_n(g;x,\varepsilon) \subset \cup\{Q \in \mathcal{Q}^{(n)} : Q \cap B_n(g;x,\varepsilon) \neq \emptyset\}$ and the cardinality of such sequence is bounded by $e^{\xi n}$ provided that ε is small. The result follows immediately. □

Let us mention that the previous estimate also holds in the case of non-additive thermodynamical formalism using the same approximation argument as above.

Acknowledgments. The authors are grateful to Y. Cao for valuable conversations. This work was initiated during a visit of the first author to the Soochow University, whose research conditions are greatly acknowledged. The first author was partially supported by CNPq and FAPESB. The second author was partially supported by NSFC(11001191) and NSF in Jiangsu Province (09KJB110007) and a Pre-research Project of Soochow University.

References

[AP06] V. Araújo and M.J. Pacífico. Large deviations for non-uniformly expanding maps. *J. Statist. Phys.*, 125:415–457, 2006.

[BCH10] J. Ban, Y. Cao and H. Hu. The dimensions of non-conformal repeller and average conformal repeller. *Trans. Amer. Math. Soc.*, 362:727–751, 2010.

[Ba96] L. Barreira. A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. *Ergodic Theory and Dynamical Systems*, 16: 871–927, (1996).

[Ba06] L. Barreira. Nonadditive thermodynamic formalism: equilibrium and Gibbs measures. *Disc. Contin. Dyn. Syst.*, 16: 279–305, 2006.

[BG06] L. Barreira, K. Gelfert Multifractal analysis for Lyapunov exponents on nonconformal repellers. *Commun. Math. Phys.*, 267: 393–418, 2006.

[Blo83] A. M. Blokh. Decomposition of dynamical systems on an interval. *Uspekhi Mat. Nauk*, 38 :179–180, 1983.

[Bow71] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–414, 1971.

[Bow74] R. Bowen. Some systems with unique equilibrium states. *Math. Systems Theory*, 8 : 193–202, 1974.

[BK83] M. Brin and A. Katok. On local entropy. *Geometric Dynamics (Rio de Janeiro) (Lecture Notes in Mathematics vol 1007)*, Spring-Verlag, Berlin-New York, 1983.

[Buz97] J. Buzzi. Specification on the interval. *Trans. Amer. Math. Soc.*, 349:2737–2754, 1997.

[Chu11] Y.M. Chung. Large deviations on Markov towers. *Nonlinearity*, 24:1229–1252, 2011.

[CFH08] Y. Cao, D. Feng and W. Huang. The thermodynamic formalism for sub-additive potentials. *Discrete Contin. Dyn. Syst.*, 20: 639–657, 2008.

[CHZ] Y. Cao, H. Hu and Y. Zhao. Nonadditive Measure-theoretic Pressure and Applications to Dimensions of an Ergodic Measure. Ergod. Th. Dynam. Syst., (to appear 2012).

[CZC12] W. Cheng, Y. Zhao and Y. Cao. Pressures for asymptotically subadditive potentials under a mistake funciton. *Discrete Contin. Dyn. Syst.*, 32(2): 487–497, 2012.

[CRL98] H. Comman and J. Rivera-Letelier. Large deviations principles for non-uniformly hyperbolic rational maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15:539–579, 1998.

[Co09] H. Comman. Strengthened large deviations for rational maps and full shifts, with unified proof. *Nonlinearity*, 22: 1413–1429, 2009.

[EKW94] A. Eizenberg, Y. Kifer and B. Weiss. Large deviations for $\mathbb{Z}^d-$actions. *Commun. Math. Phys.*, 164: 433–454, 1994.

[FL02] D.-J. Feng and K.-S. Lau. The pressure function for products of non-negative matrices. *Math. Res. Lett.*, 9: 363–378, 2002.

[FK12] D. Feng and A. Käenmäki Equilibrium states of the pressure function for products of matrices *Discrete Cont. Dyn. Sys.*, (to appear 2012)

[FH10] D. Feng, W. Huang, Lyapunov spectrum of asymptotically sub-additive potentials. *Commun. Math. Phys.*, 297: 1–43, 2010.

[IY11] G. Iommi and Y. Yayama. Almost-additive thermodynamical formalism for countable Markov shifts. *Nonlinearity*, 25: 165–191, 2012.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. IHES*, 51:137–173, 1980.

[Kif90] Y. Kifer. Large deviations in dynamical systems and stochastic processes. *Transactions of the Americal Mathematical Society*, 321(2):505–524, 1990.

[KN91] Y. Kifer and S. E. Newhouse. A global volume lemma and applications. *Israel J. Math.*, 74(2-3):209–223, 1991.

[Mel09] I. Melbourne. Large and moderate deviations for slowly mixing dynamical systems. *Proc. Amer. Math. Soc.*, 137:1735–1741, 2009.

[MN08] I. Melbourne and M. Nicol. Large deviations for nonuniformly hyperbolic systems. *Trans. Amer. Math. Soc.*, 360:6661–6676, 2008.

[MV09] A. Méson and F. Vericat. Estimates of large deviations in dynamical systems by a non-additive thermodynamic formalism. *Far East J. Dyn. Syst.*, 11: 1–16, 2009.

[Mu06] A. Mummert. The thermodynamic formalism for almost-additive sequences. *Discrete Contin. Dyn. Syst.*, 16: 435–454, 2006.

[PS05] C.-E. Pfister and W. Sullivan. Large Deviations Estimates for Dynamical Systems without the Specification Property. Application to the Beta-Shifts. *Nonlinearity*, 18, 237–261 (2005).

[PS07] C.-E. Pfister and W.G. Sullivan, On the topological entropy of saturated sets, *Ergod. Th. Dynam. Syst.*, 27:929-956, 2007.

[PS09] M. Pollicott and R. Sharp. Large deviations for intermittent maps. *Nonlinearity*, 22: 2079–2092, 2009.

[Ren57] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hung.*, 8: 477–493,1957.

[RBY08] L. Rey-Bellet and L.-S. Young. Large deviations in non-uniformly hyperbolic dynamical systems. *Ergodic Theory Dynam. Systems*, 28(2):587–612, 2008.

[RVZ12] J. Rousseau, P. Varandas and Y. Zhao. Entropy formula for dynamical systems with mistakes. *Disc. Cont. Dynam. Sys.*, (to appear 2012).

[STV03] B. Saussol, S. Troubetzkoy, and S. Vaienti. Recurrence and Lyapunov exponents. *Mosc. Math. J.*, 3:189–203, 2003.

[SSY09] K. Sakai, N. Sumi and K. Yamamoto. Diffeomorphisms satisfying the specification property. *Proc. Amer. Math. Soc.*, 138:315–321, 2009.

[Th10] D. Thompson Irregular sets, the beta-transformation and the almost specification property. Trans. Amer. Math, Soc. (to appear).

[VV10] P. Varandas and M. Viana. Existence, uniqueness and stability of equilibrium states for non-uniformly expanding maps. *Ann. I. H. Poincaré - Analyse Non-Lineaire*, 27:555–593, 2010.

[Va12] P. Varandas. Non-uniform specification and large deviations for weak Gibbs measures. *J. Statist. Phys.*, 146, 330–358, 2012.

[You90] L.-S. Young. Some large deviations for dynamical systems. *Trans. Amer. Math. Soc.*, 318:525–543, 1990.

[Yu00] M. Yuri. Weak Gibbs measures for certain nonhyperbolic systems. *Ergod. Th. and Dyn. Sys.*, 20: 1495–1518, 2000.

[Yu07] M. Yuri. Large deviations for countable to one Markov systems. *Comm. Math. Phys.*, 258:455–474, 2005.

[ZZC11] Y. Zhao, L. Zhang and Y. Cao, The asymptotically additive topological pressure on the irregular set for asymptotically additive potentials, Nonlinear Analysis, 74, (2011), 5015-5022.

Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.

E-mail address: `paulo.varandas@ufba.br, http://www.pgmat.ufba.br/varandas/`

Departament of Mathematics, Soochow University, Suzhou 215006, Jiangsu, P.R. China

E-mail address: `zhaoyun@suda.edu.cn`